

Supermanifold Forms and Integration. A Dual Theory. *

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Abstract

We investigate forms on supermanifolds defined as Lagrangians of “copaths”. For this, we consider direct products $M^{n|m} \times \mathbb{R}^{r|s}$ and study isomorphisms corresponding to simultaneously advancing the number of additional parameters $r|s$ and the number of equations. We define an exterior differential in terms of variational derivatives w.r.t. a copath and establish its main properties. In the resulting stable picture we obtain infinite complexes $\bar{d} : \Omega^{r|s} \rightarrow \Omega^{r+1|s}$ for $M^{n|m}$, where $0 \leq s \leq m$ and r can be any integer. For $r \geq 0$ a canonical isomorphism with forms constructed as Lagrangians of $r|s$ -paths is established. We discover the “lacking half” of forms on supermanifolds: $r|s$ -forms with $r < 0$, previously unknown except for $s = m$. (They have been partly replaced earlier by an augmentation of the “non-negative” part of the complexes.) The study of these questions is in progress now.

Introduction

Most of “supermathematics” can be obtained more or less easily by extending “purely even” constructions with the help of the sign rule. Supermanifold integration is not the case. Since the times when the Berezin integral was discovered and the notion of supermanifold came into being, an adequate theory of “super” forms suiting for integration has been a great puzzle. (The naive definition of differential forms in super case has nothing to do with the Berezin integration.) This was not for chance (see [11] for discussion). The construction of the theory in question took great effort in years 1975–1985. See Bernstein and Leites [2, 3], Gajduk, Khudaverdian and Schwarz [4],

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Baranov and Schwarz [1], Voronov and Zorich [12, 13, 14, 15], Voronov [10, 11], and others. An essential feature of the theory of forms developed in [12, 13, 14, 15, 10, 11] is that forms of (even) degree greater than the even dimension of a supermanifold are present necessarily.

In this paper we develop a “dual” approach, in which forms are treated as Lagrangians of “copaths” (systems of equations, which may specify nonsingular submanifolds or may not). To introduce a differential, we extend supermanifolds in consideration by introducing additional variables and study a sort of “stabilization”, reminiscent of K -theoretic methods. This leads us to a discovery of a “lacking half” of the previously developed theory of forms, namely, forms (and perhaps cohomology) of negative degree. All results are new. The study of these questions is in progress now.

(I would like to note that the “dual approach” to supermanifold integration has been independently investigated by O.M.Khudaverdian who obtained important results.)

The paper is organized as follows. In §1 we present a survey of forms as Lagrangians of paths (a “standard” theory). For greater details we refer to [11]. In §2 we introduce Lagrangians of copaths and study the variation of the corresponding action. In §3 we study “mixed Lagrangians” and prove the “stabilization theorem” (Theorem 3.1). We introduce a differential for “mixed forms” and establish its main properties (Theorem 3.2). It is shown how the complexes of “standard” forms and dual forms are sewed together in the “stable” picture.

Throughout the paper we use the notation of the book [11]. I wish to thank J.N.Bernstein, O.M.Khudaverdian and M.A.Shubin. Discussions of supermanifold forms and integration with them has stimulated this research greatly and I am much grateful to them.

§ 1. Forms as Lagrangians of multidimensional paths.

Let $M = M^{n|m}$ be a $n|m$ -dimensional supermanifold. An $r|s$ -path in M is a smooth map $\Gamma : I^{r|s} \rightarrow M$, where $I^{r|s} = I^r \times \mathbb{R}^{0|s} \subset \mathbb{R}^{r|s}$ is taken with a fixed boundary. (Fixing boundary is necessary for Berezin integration over a domain, see [11] and also below.) Denote coordinates on M by x^A , on $I^{r|s}$ by t^F , so the path Γ is presented as $x^A = x^A(t)$.

Recall that a Lagrangian function or simply a Lagrangian on M is a function on the tangent bundle TM . It depends on a position and on a single tangent vector. Integrating a Lagrangian over a path (that is, a $1|0$ -path), one obtains the action of the path. Note that in this case dx/dt is an even vector.

We shall call a function of a position and of an array of $r + s$ tangent vectors, of which r are even and s are odd, a (*generalized*) *Lagrangian on M* . We omit the word “generalized” in the following. A Lagrangian $L = L(x, \dot{x})$, where $\dot{x} = (\dot{x}_F^A)$, defines an action for $r|s$ -path:

$$S[\Gamma] = \int_{I^{r|s}} Dt L \left(x(t), \frac{\partial x}{\partial t}(t) \right). \quad (1.1)$$

Here Dt stands for a Berezin “volume element” on superspace.

Recall some facts about the Berezin integral (see [11]). Roughly speaking, Berezin integration consists in singling out the top-order coefficient in the power expansion in odd variables and in usual integration over even variables. For bounded domains a step-function $\theta(g)$ is inserted. Here the boundary is specified by the equation $g = 0$. Since the even function g may contain nilpotents, integration uses more data than just the underlying set of points of the domain. The formula for the change of variables in the usual Riemann or Lebesgue integral contains the absolute value of the Jacobian. Its analogue for the change of variables in the Berezin integral is the function $\text{Ber}_{1,0} J = \text{Ber } J \cdot \text{sign det } J_{00}$. Here J is the Jacobian matrix of the transformation of variables and J_{00} stands for its even-even block. For a block square matrix (blocks are specified by attributing “parity” to columns and rows) with even elements in the diagonal blocks and odd elements in the antidiagonal blocks the Berezinian is defined as

$$\begin{aligned} \text{Ber } J &= \det(J_{00} - J_{01}J_{11}^{-1}J_{10}) \cdot (\det J_{11})^{-1} \\ &= \det J_{00} \cdot (\det(J_{11} - J_{10}J_{00}^{-1}J_{01}))^{-1}. \end{aligned} \quad (1.2)$$

It is a unique (in essence, up to trivial changes) function on invertible matrices near unity that is multiplicative. It is essential that the general linear group in supercase possesses four, not two, connected components. They are specified by various combinations of the signs of the even-even and the odd-odd blocks. Two functions, $\det J$ and $|\det J|$, in supercase turn into four: $\text{Ber}_{\alpha\beta} J = \text{Ber } J \cdot (\text{sign det } J_{00})^\alpha \cdot (\text{sign det } J_{11})^\beta$, where $\alpha, \beta = 0, 1$. Recall that an orientation of vector space is a class of frames modulo linear transformations with positive determinant. We see that in supercase principally different notions of orientation are possible. The most important are a $(+|-)$ -orientation preserved by linear transformations with $\det J_{00} > 0$ and a $(-|-)$ -orientation preserved by transformations with $\text{Ber } J_{00} > 0$ (see [11]). We also arrive to four possible types of “volume elements” $D_{\alpha\beta}$:

$$D_{\alpha\beta}x = D_{\alpha\beta}x' \cdot \frac{D_{\alpha\beta}x}{D_{\alpha\beta}x'}, \quad \text{where} \quad \frac{D_{\alpha\beta}x}{D_{\alpha\beta}x'} = \text{Ber}_{\alpha\beta} \left(\frac{\partial x}{\partial x'} \right). \quad (1.3)$$

Here $Dx = D_{0,0}x$ and $D_{0,1}x$ are the analogues of what is called the “oriented volume element” in the classical case and $D_{1,0}x$ and $D_{1,1}x$ are the analogues of the “unoriented volume element” (denoted sometimes by $|dx|$ or similar). Actually, an *unoriented volume element* in supercase is $D_{1,0}x$. The expression $f(x)D_{1,0}x$, where f is a function, can be integrated over supermanifold without any conditions on orientation. At the same time, integration of the volume form like $f(x)Dx$ requires that the supermanifold in question should be endowed with a $(+|-)$ -orientation.

Return to the integral (1.1). Consider changes of parametrization for the path Γ .

PROPOSITION 1.1. *The action $S[\Gamma]$ is invariant under arbitrary reparametrization iff*

$$L(x, h\dot{x}) = \text{Ber}_{1,0} h \cdot L(x, \dot{x}) \quad (1.4)$$

for any $h \in \text{GL}(r|s)$.

The Lagrangians with such property are the multidimensional analogue of the homogeneous Lagrangian functions (in the classical sense) like the Lagrangian of the free particle taken in the form $m|\dot{x}|$. They are well-defined only if all vectors \dot{x}_F are linear independent, in particular no vector can vanish. That means that dealing with such Lagrangians one should consider only paths which are immersions. This would be a bad starting point for a theory of forms. Instead we need to restrict the possible reparametrization of paths by certain orientation conditions.

DEFINITION 1.1. A Lagrangian L is called *covariant (of the first kind)*, if

$$L(x, h\dot{x}) = \text{Ber } h \cdot L(x, \dot{x}), \quad (1.5)$$

and it is called *covariant (of the second kind)*, if

$$L(x, h\dot{x}) = \text{Ber}_{0,1} h \cdot L(x, \dot{x}), \quad (1.6)$$

for any $h \in \text{GL}(r|s)$.

It follows that for covariant Lagrangians of the first kind the action is invariant under any changes of parameters that preserve the $(+|-)$ -orientation of the path Γ . The same is true for covariant Lagrangians of the second kind and the $(-|-)$ -orientation. The natural widest domain of definition for covariant Lagrangians considered on some $U \subset M$ is an open domain $W_{r|s}(U) \subset \underbrace{TU \oplus \dots \oplus TU}_r \oplus \underbrace{\Pi TU \oplus \dots \oplus \Pi TU}_s$ specified by the condition $\text{rank}(\dot{x}_\alpha^\mu) = s$. (Here $\alpha = 1 \dots s$ runs over odd indices F and $\mu = 1 \dots m$ runs over odd A .) By Π we denote the parity reversion functor. We consider covariant Lagrangians defined everywhere on $W_{r|s}$.

DEFINITION 1.2. A smooth map of supermanifolds is *proper*, if it induces monomorphisms of the odd subspaces of the tangent spaces (at each point).

Obviously, the pull-back of covariant Lagrangians is well-defined provided the map is proper.

There exist nonvanishing covariant Lagrangians with $r > n$, if $m > 0$. This will be obvious if one recalls the definition of the “naive” differential forms in supercase. The algebra of such forms is generated by differentials dx^A , which have parity opposite to that of coordinates. So if $m > 0$, there exist products of $d\xi^\mu$ of arbitrary high degree. Since any “naive” differential form can be integrated over $r|0$ -paths, this provides the desired examples of covariant Lagrangians. Examples with $s > 0$ exist too (see [11]).

Consider the variational derivative of the action (1.1).

PROPOSITION 1.2.

$$\begin{aligned} \frac{\delta S}{\delta x^A} &= \frac{\partial L}{\partial x^A} - (-1)^{\tilde{A}\tilde{F}} \frac{\partial}{\partial t^F} \left(\frac{\partial L}{\partial \dot{x}_F^A} \right) \\ &= \frac{\partial L}{\partial x^A} - (-1)^{\tilde{A}\tilde{F}} \dot{x}_F^B \frac{\partial^2 L}{\partial x^B \partial \dot{x}_F^A} \\ &\quad - (-1)^{\tilde{A}\tilde{F}} \frac{1}{2} \ddot{x}_{FG}^B \left(\frac{\partial^2 L}{\partial \dot{x}_G^B \partial \dot{x}_F^A} + (-1)^{\tilde{G}\tilde{F} + \tilde{A}(\tilde{G} + \tilde{F})} \frac{\partial^2 L}{\partial \dot{x}_F^B \partial \dot{x}_G^A} \right). \end{aligned} \quad (1.7)$$

DEFINITION 1.3. A covariant Lagrangian L is called a *form* (of degree $r|s$), if the last term in (1.7) identically vanish:

$$\frac{\partial^2 L}{\partial \dot{x}_F^A \partial \dot{x}_G^B} + (-1)^{\tilde{F}\tilde{G} + \tilde{B}(\tilde{F} + \tilde{G})} \frac{\partial^2 L}{\partial \dot{x}_G^A \partial \dot{x}_F^B} = 0 \quad (1.8)$$

for any A, B, F, G .

The equations (1.8) guarantee that the variation does not depend on the “acceleration”. To proceed with the physical analogy, such Lagrangians correspond to interaction terms like $-eA_a \dot{x}^a$, for a charged particle in the electromagnetic field. One can check that in purely even case the equations (1.8) together with the covariance condition are equivalent to that L is multilinear and skew-symmetric function of tangent vectors. (In general, the equations (1.8) imply that L is skew-symmetric and multiaffine both in even rows and even columns of (\dot{x}_F^A) .) So in this case our definition reproduces the classical concept of an exterior form. The equations (1.8) are functorial, so forms can be pulled back along proper morphisms.

Forms can by no means be multilinear in odd vectors, since they should be homogeneous of degree -1 w.r.t. them. The equations (1.8) could be regarded as nontrivial substitution for multilinearity. The odd-odd part of (1.8) is a system of partial differential equations for a function of an even matrix. A remarkable fact is that this system (or similar) appears in classical integral geometry, and is typical in connection with Radon-like integral transforms (see [7, 5]). There are many unexpected links of integration on supermanifolds with classical integral geometry (see [11, 9, 15, 6]).

DEFINITION 1.4. A *differential* of $r|s$ -form L is an $(r+1|s)$ -form dL defined as follows:

$$dL = (-1)^r \dot{x}_{r+1}^A \left(\frac{\partial L}{\partial x^A} - (-1)^{\tilde{A}\tilde{F}} \dot{x}_F^B \frac{\partial^2 L}{\partial x^B \partial \dot{x}_F^A} \right). \quad (1.9)$$

One can check that the operation d is well-defined. The identity $d^2 = 0$ holds. Forms can be integrated over *singular manifolds* $f : P^{r|s} \rightarrow M^{n|m}$ (f is required to be proper), provided the orientation of the appropriate kind is fixed on $P^{r|s}$. If $P^{r|s}$ is considered together with a chosen boundary, then the Stokes formula holds:

$$\int_{(P^{r|s}, f)} dL = \int_{(Q^{r-1|s}, f \circ i)} L, \quad (1.10)$$

where L is a $(r-1|s)$ -form and $i : Q^{r-1|s} \rightarrow P^{r|s}$ the inclusion of the boundary.

The problem of the theory of supermanifold forms, in the sense of Definition 1.3, is that no explicit description is at hand. An exception is some special cases. For $s = 0$ there is 1 – 1-correspondence with the “naive” differential forms. For $s = m$ (odd codimension zero) it can be proved that $r|m$ -forms are in bijection with so called

integral forms of Bernstein and Leites [2], if $r \geq 0$ (see [11]). Note that these cases ($s = 0$ or $s = m$) are exceptional also because here the difference between forms of first and second kind is immaterial.

For forms of the second kind there is a construction supplying plenty of examples. Let $\omega = \omega(x, dx)$ be a Bernstein–Leites pseudodifferential form (p.d.f.) [3]. It is a function that need not be polynomial in dx^A . (For example, $e^{-(d\xi)^2}$ is possible.) This is a beautiful construction but one has to pay for this beauty. First, there is no grading. Second, if $\omega(x, dx)$ sufficiently rapidly decreases at the infinity of $d\xi^\mu$, it can be integrated over a singular manifold of arbitrary dimension [3], but *only* in the $(-|-)$ -oriented case. It doesn't work for $(+|-)$ -orientation. An integral of ω over (P, f) is defined as the integral of $f^*\omega$. For a $(-|-)$ -oriented supermanifold M an integral of p.d.f. ω is defined [3] as:

$$\int_M \omega := \int_{\Pi TM} Dx D(dx) \omega(x, dx). \quad (1.11)$$

(In the classical case this coincides with extracting the term of maximal degree from an inhomogeneous form and integrating it over a manifold.) This leads to the following integral transforms, which map Bernstein–Leites p.d.f.'s to $r|s$ -forms of the second kind (see [1, 13, 11]):

$$L(x, \dot{x}) = \int_{\mathbb{R}^s | r} D(dt) \omega(x, dt \cdot \dot{x}). \quad (1.12)$$

No transform like this is available for forms of the first kind.

One of the motivations for the theory of forms on supermanifolds is the problem of supermanifold cohomology. The naive differential forms as well as integral forms and pseudodifferential forms reproduce the cohomology of the underlying space. There is some hope that forms in the sense of Definition 1.3 can give richer cohomology. The actual situation is the following. From homotopy point of view, the category of supermanifolds is equivalent to the category of vector bundles [11] (with proper maps and fiberwise monomorphisms, respectively). So there is a problem of developing a “cohomology theory” for vector bundles (in the indicated category). There exists a spectral sequence analogous to the Atiyah–Hirzebruch sequence [15, 11]. Its limit term is adjoined to the cohomology of (global) forms on $M^{n|m}$ and the second term is the cohomology of the underlying space with coefficients in the cohomology of forms on $\mathbb{R}^{0|m}$. Here $\mathbb{R}^{0|m}$ plays the rôle of “point” in standard homotopy theory. A remarkable fact is that this “point cohomology” for some $s \neq 0, m$ is not trivial. There exist estimates from the below for it (for details see [15, 11]). Unfortunately, the complete calculation for “supermanifold de Rham cohomology” is not available for the present moment (see [11] for discussion).

The results on cohomology described above are seriously based on the “Cartan calculus” for $r|s$ -forms. It was found [11] that the homotopy identity (or the “Main

Formula of the Differential Calculus for Forms”) is not valid unless we introduce in a formal way the “exact 0|s-forms” $B^{0|s}$. Actually they are defined as discrepancies for the generalized homotopy identity [11]. Only after the complexes of $\cdot|s$ -forms are augmented in this way, the cohomology theory can be developed. At the other hand, the comparison with Bernstein–Leites integral forms, which are isomorphic to $r|m$ -forms, if $r \geq 0$, but which are defined and nonvanishing for $r < 0$, suggests [11] a search for forms of negative (even) degree for an arbitrary s .

§ 2. Copaths and dual Lagrangians.

Consider an open domain U in a supermanifold $M^{n|m}$. A *copath* in U is an array of functions $f^K \in C^\infty(U)$ enumerated by indices K , which run over “even” and “odd” values so that f^K can be formally treated as coordinates on some $\mathbb{R}^{p|q}$. The dimension $p|q$ is called a (formal) *codimension* of the copath. A copath can be also treated as a “framed ideal” in $C^\infty(U)$, i.e. an ideal with chosen generators. The corresponding closed subspace of U is a submanifold (of codimension $p|q$), if the functions f^K are independent.

Consider in a formal way the following integral (the “action”):

$$S[f] = \int_{U^{n|m}} Dx \delta(f) \mathcal{L} \left(x, \frac{\partial f}{\partial x} \right). \quad (2.1)$$

We leave aside here the questions of the convergence of the integral, the influence of the boundary and the existence of the delta-function $\delta(f)$.

Here $\mathcal{L} = \mathcal{L}(x, p)$ is a function of the local coordinates x^A and the components of the covectors (“momenta”) $p^K = (p_A^K)$. (Warning: one should not mix the notation for the momenta $p = (p_A^K)$ with p as the number of even functions f^K !) As a tensor object \mathcal{L} should be a component of an unoriented volume form, so to make the integral (2.1) independent on the choice of coordinates. We shall call such functions \mathcal{L} (*dual*) *Lagrangians* on M .

DEFINITION 2.1. The number of p^K and its complement are called the *codegree* and the *degree* of \mathcal{L} , respectively:

$$\text{codeg } \mathcal{L} := p|q, \quad (2.2)$$

$$\text{deg } \mathcal{L} := \dim M - \text{codeg } \mathcal{L} = n - p|m - q. \quad (2.3)$$

Consider the variation of the action (2.1). This needs certain caution. First, one can treat (2.1) as a usual action for the field f^K with a singular Lagrangian density $\Lambda(x, f, \partial f / \partial x) = \delta(f) \mathcal{L}(x, \partial f / \partial x)$. The formal calculation of the variational derivative according to the Euler–Lagrange formulae leads to

$$\delta S = \pm \int Dx Y^K \left(\frac{\partial \delta}{\partial f^K} \mathcal{L} - (-1)^{\tilde{A}\tilde{K} + (\tilde{A} + \tilde{K})\tilde{\delta}} \frac{\partial}{\partial x^A} \left(\delta \frac{\partial \mathcal{L}}{\partial p_A^K} \right) \right), \quad (2.4)$$

where $Y = (Y^K)$ is the variation of f : $f^K \mapsto f^K + \varepsilon Y^K$, $\varepsilon^2 = 0$. (The common sign in (2.4) depends on dimensions and the parity of Y and is irrelevant.) Here the substitution $f^K = f^K(x)$, $p_A^K = \partial f^K / \partial x^A$ before taking the “total” derivative by x^A is supposed, as usual.

One should note, however, that this variation of S corresponds only to a limited class of variations of f^K , namely to the variations compactly supported in U . If we are going to consider our “copaths” as corresponding to surfaces in M , then we should also consider variations which correspond to the change of equations $f^K = 0$ for an equivalent system. These variations of f^K are of the form $Y^K = f^L Z_L^K$, where arbitrary $Z_L^K \in C^\infty(U)$ are not supposed to be compactly supported.

PROPOSITION 2.1. *The variation of the action is*

$$\delta S = \pm \int Dx Z_L^K \delta(f) \left(-(-1)^{\tilde{K}} \delta_K^L \mathcal{L} + (-1)^{\tilde{A}(\tilde{K}+\tilde{L})} p_A^L \frac{\partial \mathcal{L}}{\partial p_A^K} \right) \quad (2.5)$$

in this case.

PROOF. Straightforward calculation, using the easily obtained identity $f^L \partial \delta / \partial f^K = -(-1)^{\tilde{K}\tilde{L}} \delta_L^K \delta(f)$. \square

Let the variation (2.5) vanish identically. This implies the equation

$$(-1)^{\tilde{A}(\tilde{K}+\tilde{L})} p_A^L \frac{\partial \mathcal{L}}{\partial p_A^K} = -(-1)^{\tilde{K}} \delta_K^L \mathcal{L} \quad (2.6)$$

for Lagrangians in question.

PROPOSITION 2.2. *The identity (2.6) is the infinitesimal form of the condition:*

$$\mathcal{L}(p \cdot g) = \mathcal{L}(p) \cdot \text{Ber } g, \quad (2.7)$$

for any matrices g taking values in the identity component of the supergroup $\text{GL}(p|q)$.

PROOF. Consider elementary transformations and multiplication by e^λ for the columns of (p_A^K) . \square

This suggests two possible conditions for the behaviour of \mathcal{L} under generic transformations of the argument p : either they multiply it by $\text{Ber } g$ or by $\text{Ber}_{0,1} g$. (Compare with two kinds of forms in the “standard” theory.)

DEFINITION 2.2. The Lagrangians satisfying (2.7) (or the similar condition with $\text{Ber}_{0,1}$) for generic g are called *covariant*. The cases of Ber and $\text{Ber}_{0,1}$ will be distinguished as covariance of the *first* or *second* kind.

For $p|q \leq n|m$ the covariant dual Lagrangians suit for integration over $p|q$ -codimensional submanifolds. If a cooriented submanifold $P \subset M$ is specified by (nondegenerate) copaths f_α for a covering (U_α) , then one can define an integral over P as the

sum of integrals over all f_α , using a partition of unity. This is well-defined due to covariance. Actually, the covariant Lagrangians of the first kind are good for $(+|-)$ -coorientation (see [11]) and those of the second kind for $(-|-)$ -coorientation. If one prefers to consider oriented submanifolds rather than cooriented, then it is necessary to take dual Lagrangians with local coefficients (namely, $\text{sign det } TM_0$ for Lagrangians of the first kind and $\text{sign Ber } TM$ for the second kind).

The covariant Lagrangians are well-defined only, if the rank of the matrix $(p_A^K)_{\tilde{K}=1}$ equals $0|q$. In particular, the odd codegree must be less than the odd dimension of the supermanifold: $q \leq m$.

From now on we demand that the odd equations of any copath are independent, i.e. the set of gradients df^K , $\tilde{K} = 1$, has rank $0|q$. At the same time, the independence of even equations is not required. On $M^{n|m}$ their number may very well be greater than n . Such copaths do not correspond to submanifolds. For dependent even f^K the integral (2.1) is senseless, since the delta-function $\delta(f)$ is not well-defined.

EXAMPLE 2.1. Let $\sigma = \sigma(x, \theta) Dx$ be a Bernstein–Leites integral form with local coefficients in the sheaf $\text{sign det } TM_0$. Let $\sigma(x, \theta)$ be a polynomial in θ_A of degree k . Set

$$\mathcal{L}(x, p) = \int_{\mathbb{R}^{0|p}} D\varphi \sigma(x, p \cdot \varphi), \quad (2.8)$$

which is well-defined for any number p . The integral vanishes for $p \neq k$. For $p = k$ this integral transform establishes an isomorphism between “exterior polynomials” σ and skew-symmetric multilinear functions of p^1, \dots, p^k . Note that if $m > 0$, these functions do not vanish for $k > n$.

We see that the covariant dual Lagrangians of codegree $p|0$ are in $1 - 1$ -correspondence with the Bernstein–Leites integral forms of degree $n - p$. In particular, we obtain an example of non-vanishing Lagrangians of negative even degree $n - p$. Although the Lagrangians of negative degree cannot be integrated (like integral forms of negative degree), they constitute an important part of the theory.

THEOREM 2.1. *For covariant Lagrangians \mathcal{L} the Euler–Lagrange equations for the action (2.1) are equivalent to*

$$(-1)^{\tilde{A}\tilde{K}} \frac{\partial}{\partial x^{\tilde{A}}} \left(\frac{\partial \mathcal{L}}{\partial p_A^K}(x, \frac{\partial f}{\partial x}) \right) \equiv 0 \pmod{(f)} \quad (2.9)$$

(i.e., the left-hand side vanishes on the surface $f^K = 0$).

PROOF. Expand the equation (2.4) and substitute (2.6) into it. Then the terms with the derivative of the delta-function cancel and we obtain

$$\delta S = \pm \int Dx Y^K \delta(f) (-1)^{\delta\tilde{K}} \left((-1)^{\tilde{K}\tilde{A}} \frac{\partial}{\partial x^{\tilde{A}}} \left(\frac{\partial \mathcal{L}}{\partial p_A^K}(x, \frac{\partial f}{\partial x}) \right) \right), \quad (2.10)$$

for an arbitrary finite variation Y . Hence if $\delta S = 0$, the expression in the outer brackets annihilates $\delta(f)$. That means that it belongs to the ideal generated by f^K . \square

DEFINITION 2.3. We call a covariant dual Lagrangian \mathcal{L} *closed*, if

$$(-1)^{\bar{A}\bar{K}} \frac{\partial}{\partial x^A} \left(\frac{\partial \mathcal{L}}{\partial p_A^K} \left(x, \frac{\partial f}{\partial x} \right) \right) = 0 \quad (2.11)$$

for any K .

That means that the action (2.1), if it makes sense, is identically stationary for any copath. (The equation (2.11) is due to O.M.Khudaverdian [8]. He introduced it for “dual densities” in the sense of [4, 8].)

Following the analogy with the “standard” theory of the previous section, one may wish to define a differential for dual Lagrangians (of degree one w.r.t. Definition 2.1), on the basis of the variation formula (2.10). Advancing the degree of \mathcal{L} (or the formal dimension of the copaths), in our dual language means reducing the number of functions f^K . Unfortunately, this is not suggested by (2.10). But, euristically, variation of any object means introducing an additional degree of freedom, corresponding to the variation parameter. This suggests an approach to the formula (2.10) by enlarging the original “configuration space” $M^{n|m}$. To do so, we are going now to introduce a sort of “mixed” theory, combining the “dual” approach with the theory of the Lagrangians of paths of the previous section.

§ 3. Mixed forms.

Consider a direct product $M^{n|m} \times \mathbb{R}^{r|s}$ for some $r|s$, with direct product coordinates on it. Let t^F denote coordinates on $\mathbb{R}^{r|s}$. Consider copaths (f^K) on $M^{n|m} \times \mathbb{R}^{r|s}$. Two particular cases are $(x^A - x^A(t))$, which are equivalent to paths in M , and $(t - t(x), f^*(x))$, which correspond to copaths $f^*(x)$ on M . As above, in a formal way consider an integral

$$S[f] = \int_{U^{n|m} \times \mathbb{R}^{r|s}} Dx Dt \delta(f) \mathcal{L} \left(x, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial t} \right). \quad (3.1)$$

There is a slight but important difference between the integrals (3.1) and (2.1) with M substituted by $M \times \mathbb{R}^{r|s}$. Here in (3.1) the function $\mathcal{L}(x, p, w)$ does not depend on t^F . That means that \mathcal{L} cannot be treated as a geometrical object (a dual Lagrangian) w.r.t. changes of both x^A , t^F . Instead, now the functional S also depends on the coordinate system on $\mathbb{R}^{r|s}$. We shall demand that \mathcal{L} behaves as dual Lagrangian w.r.t. (x^A) . So the integral (3.1) does not depend on the choice of coordinates on M . We call functions \mathcal{L} *mixed Lagrangians* on M . The number $p|q$ is called a *codegree*: codeg \mathcal{L} , and $r|s$ is

called an *additional degree*: $\text{add. deg } \mathcal{L}$. The *degree* of \mathcal{L} is naturally calculated as

$$\begin{aligned} \deg \mathcal{L} &= \dim M + \text{add. deg } \mathcal{L} - \text{codeg } \mathcal{L} \\ &= n + r - p \mid m + s - q. \end{aligned} \quad (3.2)$$

(As before, we consider Lagrangians of any degree, although the integral (3.1) has sense only for $\deg \mathcal{L} \geq 0$).

EXAMPLE 3.1. Let a $(-|-)$ -oriented closed submanifold $P \subset M$ be specified (locally) by a “mixed” system of equations $f^K(x, t) = 0$. Then for a Bernstein–Leites pseudodifferential form $\omega = \omega(x, dx)$

$$\int_P \omega = \pm \int Dx D(dx) Dt D(dt) \delta(f) \delta(df) \omega(x, dx). \quad (3.3)$$

Indeed, the right-hand side is independent on the choices of coordinates and equations (compatible with the orientation) and coincides with the left-hand side after elimination of t .

EXAMPLE 3.2. Analogously, for a (pseudo)integral Bernstein–Leites form $\sigma = \sigma(x, \theta) Dx$ with local coefficients in the “orientation sheaf” $\text{sign det } TM_0$

$$\int_P \sigma = \pm \int Dx Dt D\varphi \delta(f) \delta\left(\frac{\partial f}{\partial t} \varphi\right) \sigma\left(x, \frac{\partial f}{\partial x} \varphi\right), \quad (3.4)$$

where φ runs over $\mathbb{R}^{q|p}$. (A possible proof is to reduce (3.4) to (3.3) by Fourier–Hodge integral expansion of σ .)

We stress that there is no analogue of the integrals (3.3) and (3.4) in the case of $(+|-)$ -orientation.

The class of arbitrary mixed Lagrangians is too wide. Indeed, if we specify a submanifold by mixed copaths, the integral (3.1) should not depend neither on coordinates on $\mathbb{R}^{r|s}$ nor on the choice of equations (up to signs related to orientation). And if we eliminate t^F from $(f^K(x, t))$, obtaining a copath of the form $(t - t(x), f^*(x))$ on $M^{n|m} \times \mathbb{R}^{r|s}$, then only $f^*(x)$ is relevant for any reasonable “integration object that lives on M ”. Naturally, the mixed Lagrangians supplied by model examples above meet these additional requirements.

DEFINITION 3.1. A mixed Lagrangian $\mathcal{L} = \mathcal{L}(x, p, w)$ is *right-covariant*, if

$$\mathcal{L}(x, pg, wg) = \mathcal{L}(x, p, w) \cdot \text{Ber } g, \quad (3.5)$$

is *left-covariant*, if

$$\mathcal{L}(x, p, hw) = \text{Ber } h \cdot \mathcal{L}(x, p, w) \quad (3.6)$$

and is *admissible*, if

$$\mathcal{L}(x, p + aw, w) = \mathcal{L}(x, p, w). \quad (3.7)$$

Here $g \in \text{GL}(p|q)$, $h \in \text{GL}(r|s)$, $a \in \text{Mat}(n|m \times r|s)$ are arbitrary. (As in Definitions 1.1, 2.2, variants of (3.5) and (3.6) with Ber or $\text{Ber}_{0,1}$ are possible and they will be distinguished, where necessary, in the same way, by the words “first” or “second” kind.)

The infinitesimal versions can be written for all (3.5), (3.6), (3.7). For example, the equation

$$w_F^K \frac{\partial \mathcal{L}}{\partial p_A^K} = 0 \quad (3.8)$$

is equivalent to the admissibility.

PROPOSITION 3.1. *If \mathcal{L} is admissible, then the action $S[f]$ of the copath $(t-t(x), f^*(x))$ does not depend on the function $t(x)$.*

PROOF. Substitute $p_A^G = -\partial t^G / \partial x^A(x)$, $p_A^K = \partial f^{*K} / \partial x^A(x)$, $w_F^G = \delta_F^G$, $w_F^K = 0$, where the superscript K runs over the equations “on M ”. Using the invariance property (3.7), we can eliminate the dependence on $\partial t^G / \partial x^A(x)$ without altering p_A^K . \square

Now let’s turn to the variation of the action (3.1). If \mathcal{L} is right-covariant, then acting formally as above for (2.1) we immediately obtain that

$$\delta S = \pm \int Dx Dt \delta(f) Y^K \left((-1)^{\tilde{K}\tilde{A}} \frac{\partial}{\partial x^A} \left(\frac{\partial \mathcal{L}}{\partial p_A^K} \right) + (-1)^{\tilde{K}\tilde{F}} \frac{\partial}{\partial t^F} \left(\frac{\partial \mathcal{L}}{\partial w_F^K} \right) \right). \quad (3.9)$$

Note that (3.9), (2.10) both include the second derivatives of f^K . The condition for δS not to depend on second derivatives is the following equations:

$$\frac{\partial^2 \mathcal{L}}{\partial p_A^K \partial p_B^L} + (-1)^{\tilde{A}\tilde{B}+(\tilde{A}+\tilde{B})\tilde{L}} \frac{\partial^2 \mathcal{L}}{\partial p_B^K \partial p_A^L} = 0, \quad (3.10)$$

$$\frac{\partial^2 \mathcal{L}}{\partial p_A^K \partial w_F^L} + (-1)^{\tilde{A}\tilde{F}+(\tilde{A}+\tilde{F})\tilde{L}} \frac{\partial^2 \mathcal{L}}{\partial w_F^K \partial p_A^L} = 0, \quad (3.11)$$

$$\frac{\partial^2 \mathcal{L}}{\partial w_F^K \partial w_G^L} + (-1)^{\tilde{F}\tilde{G}+(\tilde{F}+\tilde{G})\tilde{L}} \frac{\partial^2 \mathcal{L}}{\partial w_G^K \partial w_F^L} = 0, \quad (3.12)$$

which are analogous to (1.8). Equation (3.10) was suggested in [8] for “dual densities”. Like for (1.8), the odd-odd part of these equations reminds equations that are familiar in classical integral geometry [7, 5]. Actually, from the abstract point of view all the equations (1.8), (3.10), (3.11), (3.12) have the same structure. They are distinguished only by the notation for independent variables. We shall call equations (1.8), (3.10–3.12) and similar the *fundamental equations*. As we shall show in a separate paper, these equations have a homological interpretation.

Let us discuss those properties of mixed Lagrangians which are connected with the elimination of variables from copaths. This may be treated as a pure algebraic problem concerning abstract functions of matrices with given conditions on their dependence on columns and rows. Consider a matrix with a chosen row and a chosen column of the same parity α . Denote their common element by u , other elements of a chosen row by w^K and of a chosen column by v_A , and all other matrix entries by p_A^K . The fundamental equations and various properties from Definition 3.1 have natural sense for functions of p, w, v, u . We'll refine only the notion of admissibility, saying that a function of a matrix is *admissible w.r.t. a given row*, if it is invariant under elementary transformations of other rows by the given one.

LEMMA 3.1. Take $\mathcal{L} = \mathcal{L}(p, w, v, u)$ and define the function $\mathcal{L}^* = \mathcal{L}^*(p, v)$ by the formula

$$\mathcal{L}^*(p, v) := \mathcal{L}(p, 0, v, 1). \quad (3.13)$$

Conversely, take $\mathcal{L}^* = \mathcal{L}^*(p, v)$ and define $\mathcal{L} = \mathcal{L}(p, w, v, u)$ as follows:

$$\mathcal{L} = \mathcal{L}(p, w, v, u) := u^{(-1)^\alpha} \mathcal{L}^*(p - vw/u, v/u). \quad (3.14)$$

In the domain where u is invertible the equations (3.13) and (3.14) supply mutually inverse isomorphisms of the spaces of right-covariant functions of matrices (p, w, v, u) and (p, v) . For right-covariant \mathcal{L} and \mathcal{L}^* the following properties of \mathcal{L} are equivalent to the properties of \mathcal{L}^* :

- (a) \mathcal{L} is admissible and homogeneous of degree $(-1)^\alpha$ w.r.t. the chosen row (w, u) . \mathcal{L}^* does not depend on v .
- (b) \mathcal{L} is left-covariant for some of the rows (p_A, v_A) . \mathcal{L}^* is left-covariant for the same rows.
- (c) \mathcal{L} is admissible w.r.t. a row (p_A, v_A) . \mathcal{L}^* is admissible w.r.t. the same row.
- (d) \mathcal{L} satisfies the fundamental equations w.r.t. the variables p_A^K, w^K, v_A, u . \mathcal{L}^* satisfies the fundamental equations w.r.t. p_A^K, v_A .

We omit the proof of Lemma 3.1 because of the lack of room. Of course, nontrivial are the “converse” statements, i.e. passing back from \mathcal{L}^* to \mathcal{L} . Although our proof includes rather long calculations, they seem quite beautiful. Especially amazing is how the fundamental equations reproduce themselves after elimination of some independent variables using covariance conditions. (We have discovered this fact experimentally, in special cases, about ten years ago, investigating the system (1.8).)

REMARKS. 1. The formula (3.14) is written under assumption of right-covariance of the first kind. In the case of the second kind it is necessary to replace $u^{(-1)^\alpha} = u^{-1}$ (for $\alpha = 1$) in (3.14) by $|u|^{-1}$.

2. Suppose $\mathcal{L} = \mathcal{L}(p, w)$ is admissible. It follows from Lemma 3.1 that right- and left-covariance conditions are compatible for \mathcal{L} , only if they are of the same kind. That

means that (3.5) and (3.6) may include either $\text{Ber } g$ and $\text{Ber } h$ or $\text{Ber}_{0,1} g$ and $\text{Ber}_{0,1} h$ but no other combination. (Else we come to a contradiction.)

DEFINITION 3.2. We call a right-covariant Lagrangian $\mathcal{L} = \mathcal{L}(x, p, w)$ a *mixed form* on M , if it is left-covariant and admissible and satisfies the equations (3.10–3.12). If $r|s = 0$, then $\mathcal{L} = \mathcal{L}(x, p)$ is called a *dual form*.

Naturally distinguished are mixed forms of first and second kind. Since for a mixed form $\mathcal{L}(p, w)$ both the odd momenta (p^K, w^K) and the odd “additional velocities” w_F must be linear independent, the following inequality holds:

$$s \leq q \leq m + s, \quad (3.15)$$

where $\text{codeg } \mathcal{L} = p|q$, $\text{add. deg } \mathcal{L} = r|s$, $\dim M = n|m$. At the same time, the even degree of \mathcal{L} , which is $n + r - p$, can be any integer, positive or negative.

THEOREM 3.1. (a) *Let $k, l \geq 0$. The space of mixed forms of additional degree $r + k|s + l$ and codegree $p + k|q + l$ is naturally isomorphic to the space of mixed forms of additional degree $r|s$ and codegree $p|q$.*

(b) *The space of mixed forms of additional degree $r|s$ and codegree $n|m$, where $n|m = \dim M$, is naturally isomorphic to the space of forms in the sense of Definition 1.3 of degree $r|s$, “twisted” by local coefficients $\text{sign det } TM_0$ (for the first kind) or $\text{sign Ber } TM$ (for the second kind).*

PROOF. (a) Consider a mixed form \mathcal{L} of codegree $p + k|q + l$ and additional degree $r + k|s + l$. We shall write the arguments of forms as matrices. Let $\begin{pmatrix} p \\ w \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$. Here we distinguish the last $k|l$ of $p + k|q + l$ momenta (written as columns) and the last $k|l$ of $r + k|s + l$ “additional velocities” (the last $k|l$ rows). For $\mathcal{L} = \mathcal{L}\left(\begin{pmatrix} p \\ w \end{pmatrix}\right)$, set $\mathcal{L}^*\left(\begin{pmatrix} p^* \\ w^* \end{pmatrix}\right) := \mathcal{L}\left(\begin{pmatrix} p^* & 0 \\ w^* & 0 \\ 0 & 1 \end{pmatrix}\right)$. Obviously, \mathcal{L}^* is a mixed form of codegree $p|q$ and additional degree $r|s$. In the open domain where the submatrix w_{22} is invertible, the form \mathcal{L} can be expressed in terms of \mathcal{L}^* as follows:

$$\mathcal{L}\left(\begin{pmatrix} p \\ w \end{pmatrix}\right) = \mathcal{L}^*\left(\begin{pmatrix} p_1 - p_2 w_{22}^{-1} w_{21} \\ w_{11} - w_{12} w_{22}^{-1} w_{21} \end{pmatrix}\right) \cdot \text{Ber}' w_{22}. \quad (3.16)$$

Here Ber' stands for Ber or $\text{Ber}_{0,1}$ for forms of first and second kind respectively. (Use right-covariance and then admissibility or left-covariance and then admissibility and right-covariance.) Consider the formula (3.16) for arbitrary mixed forms \mathcal{L}^* . The transformations of the arguments that we consider can be reduced to steps, to any of

which Lemma 3.1 is applicable. Hence, the function \mathcal{L} , defined in the domain where w_{22}^{-1} exists, possesses all properties of a mixed form. It can be shown that \mathcal{L} uniquely extends to all possible values of its arguments. First, by the skew-symmetry w.r.t. both the momenta and the additional velocities of given parity (which follows from the covariance conditions), \mathcal{L} uniquely extends to the domain where *some* $(k|l \times k|l)$ -submatrix of w , not necessarily w_{22} , is invertible. That is, to the domain $\text{rank } w \geq k|l$. Since the rank of the odd rows of w is $s+l \geq l$, this is a condition only on the even-even block of w , which has dimension $r+k \times p+k$. The fundamental equations imply that \mathcal{L} is multilinear both in the even momenta and the even additional velocities. Thus, as a polynomial, it is uniquely determined by its restriction to an open domain. So we have proved that the maps $\mathcal{L} \mapsto \mathcal{L}^*$ and $\mathcal{L}^* \mapsto \mathcal{L}$ are indeed the desired mutually inverse isomorphisms. Their coordinate independence is checked immediately.

(b) Let $\mathcal{L} = \mathcal{L} \left(\begin{smallmatrix} p \\ w \end{smallmatrix} \right)$ be a mixed form of codegree $n|m$. Define $L(\dot{x}) := \mathcal{L} \left(\begin{smallmatrix} 1 \\ \dot{x} \end{smallmatrix} \right)$.

In other coordinate system it would be $L'(\dot{x}') = L'(\dot{x} \partial x' / \partial x) = \mathcal{L}' \left(\begin{smallmatrix} 1 \\ \dot{x} \partial x' / \partial x \end{smallmatrix} \right) = \mathcal{L} \left(\begin{smallmatrix} \partial x' / \partial x \\ \dot{x} \partial x' / \partial x \end{smallmatrix} \right) \cdot D_{1,0}x / D_{1,0}x' = \mathcal{L} \left(\begin{smallmatrix} 1 \\ \dot{x} \end{smallmatrix} \right) \cdot \text{Ber}' \partial x' / \partial x \cdot D_{1,0}x / D_{1,0}x' = \pm L(\dot{x})$, where \pm equals $\text{sign det}(\partial x' / \partial x)_{00}$ or $\text{sign Ber } \partial x' / \partial x$ for the first or the second kind, respectively. Obviously, L is covariant and satisfies the fundamental equations. Conversely, take any form L , twisted by the appropriate local coefficients, and set $\mathcal{L} \left(\begin{smallmatrix} p \\ w \end{smallmatrix} \right) := L(wp^{-1}) \cdot \text{Ber}' p$, in the domain where p is invertible. From Lemma 3.1 it follows that \mathcal{L} is a mixed form. And, as above, \mathcal{L} uniquely extends to all values of arguments. \square

The isomorphisms of Theorem 3.1 are compatible with integration. For a submanifold $P \subset M$ represented by mixed copaths, the integral of a mixed form \mathcal{L} over P , which is locally expressed by an integral over a copath, after the elimination of any of the parameters t^F becomes an integral of the corresponding \mathcal{L}^* over the new copath of less codimension. The same is true if one eliminates the coordinates x^A in order to get an explicit local parametrization of $P \subset M$. Then the integral of \mathcal{L} over P becomes an integral of the corresponding L .

Now we are ready to introduce a differential for mixed forms. First, let \mathcal{L} be an arbitrary Lagrangian of codegree $p|q$ and additional degree $r|s$.

DEFINITION 3.3. Define its *differential* $d\mathcal{L}$ as follows:

$$d\mathcal{L} := (-1)^r w_{r+1}^K \left((-1)^{\tilde{A}\tilde{K}} \frac{\partial}{\partial x^A} \left(\frac{\partial \mathcal{L}}{\partial p_A^K} \right) + (-1)^{\tilde{F}\tilde{K}} \frac{\partial}{\partial t^F} \left(\frac{\partial \mathcal{L}}{\partial w_F^K} \right) \right). \quad (3.17)$$

(The sign $(-1)^r$ is inserted for convenience.)

Here $d\mathcal{L}$ depends on extra even “additional velocity” w_{r+1} . Since the formula (3.17)

includes the total differentiation w.r.t. x^A and t^F , the function $\bar{d}\mathcal{L}$ also depends on an array of new variables $\ddot{f}_{AB}, \ddot{f}_{AF}, \ddot{f}_{FG}$, unless (3.10–3.12) are satisfied.

If \mathcal{L} is a form, the formula for $\bar{d}\mathcal{L}$ is simplified:

$$\bar{d}\mathcal{L} = (-1)^r w_{r+1}^K (-1)^{\bar{A}\bar{K}} \frac{\partial}{\partial x^{\bar{A}}} \frac{\partial \mathcal{L}}{\partial p_A^K} \quad (3.18)$$

(because there is no explicit dependence on t).

THEOREM 3.2. *The definition of \bar{d} does not depend on the choice of coordinates. If \mathcal{L} is a form, so is $\bar{d}\mathcal{L}$. The identity $\bar{d}^2 = 0$ holds. The differential \bar{d} commutes with the isomorphisms (a) and (b) of Theorem 3.1.*

PROOF OF THEOREM 3.2. Consider a change of coordinates on M : $x^A = x^A(x')$. Since $\mathcal{L}(x, p, w)$ is a volume form component (a “scalar density of tensor weight one”), its derivative w.r.t. a component of a momentum $\mathfrak{X}_K^A = (-1)^{\bar{A}(\bar{\mathcal{L}}+1)} \partial \mathcal{L} / \partial p_A^K$ behaves as a vector density of weight one (for any K). So the divergence $\text{div } \mathfrak{X}_K = (-1)^{\bar{A}(\bar{\mathfrak{X}}_K+1)} \partial \mathfrak{X}_K^A / \partial x^{\bar{A}}$ is a well-defined scalar density. This is exactly the first term inside the brackets in (3.17). As for the second term, its tensor behaviour is obviously the same as that of \mathcal{L} . Thus we proved that the operation \bar{d} is well-defined for any Lagrangian \mathcal{L} . It is convenient to divide the remaining proof into the following Lemmas. Let \mathcal{L} be a mixed form. Note that below the indices F, G and similar enumerate the “old” additional velocities w_F , which do not include w_{r+1} .

LEMMA 3.2. *$\bar{d}\mathcal{L}$ is right-covariant.*

PROOF. Consider the linear transformations of the momenta $(p_A^K, w_F^K, w_{r+1}^K)$, the arguments of $\bar{d}\mathcal{L}$. The components (w_{r+1}^K) and the partial derivatives $\partial / \partial p_A^K$ transform contragrediently. So the transformations reduce to transformations in the argument of the form \mathcal{L} and we can apply the right-covariance condition. \square

LEMMA 3.3. *$\bar{d}\mathcal{L}$ is left-covariant.*

PROOF. Consider the linear transformations of the “velocities” (w_F, w_{r+1}) . The transformations among (w_F) obviously commute with $\partial^2 / \partial x^{\bar{A}} \partial p_A^K$, so we can apply the left-covariance of \mathcal{L} . The homogeneity in w_{r+1} is evident. Consider an elementary transformation of the form $w_{r+1} \mapsto w_{r+1} + \varepsilon w_F$. It leaves $\bar{d}\mathcal{L}$ invariant due to the admissibility of \mathcal{L} . Finally, consider the elementary transformation $w_F \mapsto w_F + \varepsilon w_{r+1}$. It leaves $\bar{d}\mathcal{L}$ invariant because of the equation (3.11). \square

LEMMA 3.4. *$\bar{d}\mathcal{L}$ is admissible.*

PROOF. Consider an elementary transformation of the row p_A by the row w_F . It obviously commutes with \bar{d} and we can apply the admissibility of \mathcal{L} . Now consider elementary transformations by w_{r+1} . They leave $\bar{d}\mathcal{L}$ invariant because of the equation (3.10). \square

Since for a mixed form \mathcal{L} the Lagrangian $\bar{d}\mathcal{L}$ does not depend on second derivatives, we may again apply to it the operation \bar{d} by the formula (3.17) and obtain a mixed Lagrangian $\bar{d}\bar{d}\mathcal{L}$.

LEMMA 3.5. $\bar{d}\bar{d}\mathcal{L} = 0$.

PROOF. We need to calculate the expression

$$w_{r+2}^L \left((-1)^{\bar{B}\bar{L}} \frac{\partial}{\partial x^B} \frac{\partial}{\partial p_B^L} + (-1)^{\bar{G}\bar{L}} \frac{\partial}{\partial t^G} \frac{\partial}{\partial w_G^L} + \frac{\partial}{\partial t^{r+1}} \frac{\partial}{\partial w_{r+1}^K} \right) \left(w_{r+1}^K (-1)^{\bar{A}\bar{K}} \frac{\partial}{\partial x^A} \frac{\partial \mathcal{L}}{\partial p_A^K} \right), \quad (3.19)$$

where the differentiation in the outer brackets is “total”. By straightforward but rather tedious calculation, (3.19) reduces to a sum of six terms, each of which vanishes by the equations (3.10–3.12). \square

Lemma 3.5 in particular implies that $\bar{d}\mathcal{L}$ satisfies the fundamental equations. Thus, by Lemmas 3.2, 3.3, 3.4 and 3.5, \bar{d} indeed maps mixed forms to mixed forms and, by Lemma 3.5, $\bar{d}^2 = 0$ on mixed forms.

Now let’s prove that \bar{d} commutes with the isomorphisms of Theorem 3.1. Let \mathcal{L}^* be a mixed form of codegree $p|q$ and additional degree $r|s$. Consider $\mathcal{L}(x, p, w) = \mathcal{L}^*(x, p_1 - p_2 w_{22}^{-1} w_{21}, w_{11} - w_{12} w_{22}^{-1} w_{21}) \cdot \text{Ber}' w_{22}$ (notation of Theorem 3.1 except that, for typographical reasons, we are not consistent in writing matrices). The codegree and additional degree for \mathcal{L} is advanced by $k|l$. Apply the operator \bar{d} to \mathcal{L} . By immediate check, $(-1)^{r+k} \bar{d}\mathcal{L}$ is expressed in terms of $(-1)^r \bar{d}\mathcal{L}^*$ by the same formula as \mathcal{L} is expressed in terms of \mathcal{L}^* . This coincides with the desired isomorphism up to the sign $(-1)^k$, because one needs to transpose w_{r+k+1} with the last k even rows w_F (in the argument of $\bar{d}\mathcal{L}$). Thus the signs cancel. The proof for the isomorphism (b) is similar. END OF THE PROOF OF THEOREM 3.2.

Denote the space of mixed forms of first or second kind twisted by $\text{sign det } TM_0$ or $\text{sign Ber } TM$, respectively, by $\Omega_{p|q}^{r|s}$. Since the author has problems with drawing in 4-dimensional space, below the numbers q and s are fixed. From Theorem 3.2 we obtain the following commutative diagram.

$$\begin{array}{ccccccccccc}
\dots & & \dots & & \dots & & \dots & & \dots & & \dots \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \Omega_{2|q}^{0|s} & \xrightarrow{d} & \Omega_{2|q}^{1|s} & \xrightarrow{d} & \Omega_{2|q}^{2|s} & \xrightarrow{d} & \Omega_{2|q}^{3|s} & \xrightarrow{d} & \Omega_{2|q}^{4|s} & \longrightarrow \dots \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
& & 0 & \longrightarrow & \Omega_{1|q}^{0|s} & \xrightarrow{d} & \Omega_{1|q}^{1|s} & \xrightarrow{d} & \Omega_{1|q}^{2|s} & \xrightarrow{d} & \Omega_{1|q}^{3|s} & \longrightarrow \dots \\
& & & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
& & & & 0 & \longrightarrow & \Omega_{0|q}^{0|s} & \xrightarrow{d} & \Omega_{0|q}^{1|s} & \xrightarrow{d} & \Omega_{1|q}^{2|s} & \longrightarrow \dots \\
\dots & & n-2 & & n-1 & & n & & n+1 & & n+2 & \dots
\end{array}$$

Here the vertical arrows are the isomorphisms from Theorem 3.1. The numbers in the line below the diagram are equal to the even degrees of mixed forms in the corresponding column. (The odd degree $m + s - q$ is the same for the whole diagram.) Let's denote mixed forms without any twist by $\Sigma_{p|q}^{r|s}$. The same diagram can be drawn for them, too. It is natural to pass to a direct limit (which is very simple here).

DEFINITION 3.4. Let $r \in \mathbb{Z}$ be any integer and let $0 \leq s \leq m$. Define the spaces of “stable” forms by

$$\begin{aligned}
\Omega^{r|s} &:= \varinjlim_{N,M} \Omega_{n-r+N|m-s+M}^{N|M}, \\
\Sigma^{r|s} &:= \varinjlim_{N,M} \Sigma_{n-r+N|m-s+M}^{N|M}.
\end{aligned} \tag{3.20}$$

Obviously,

$$\begin{aligned}
\Omega^{r|s} &= \Omega_{n-r+N|m-s+M}^{N|M}, \\
\Sigma^{r|s} &= \Sigma_{n-r+N|m-s+M}^{N|M},
\end{aligned} \tag{3.21}$$

for $N \geq 0$, $N \geq r - n$, $M \geq 0$, $M \geq s - m$. In particular, if $r \geq 0$, then

$$\Omega^{r|s} = \Omega_{n|m}^{r|s} = \Omega^{r|s} \tag{3.22}$$

(forms in the sense of § 1). And if $r \leq n$, then

$$\Omega^{r|s} = \Omega_{n-r|m-s}^{0|0} \tag{3.23}$$

(twisted dual forms). Thus the “stable complexes” $\Omega^{\cdot|s}$ can be completely described in terms of $\Omega^{\cdot|s}$ and $\Omega_{\cdot|q} := \Omega_{\cdot|q}^{0|0}$. The differential on $\Omega^{\cdot|s}$ induce the following differential for dual forms:

$$\delta \mathcal{L} := (-1)^{k-1} \frac{\partial}{\partial x^A} \frac{\partial \mathcal{L}}{\partial p_A^k}, \tag{3.24}$$

for $\mathcal{L} \in \Omega_{k|q}$. (It follows from Theorems 3.1, 3.2 that \mathcal{L} is closed, iff $\delta\mathcal{L} = 0$.) The complexes $\Omega^{|\cdot|s}$ and $\Omega_{|\cdot|m-s}$ are sewed together as follows:

$$\begin{array}{ccccccccccccccc}
 0 & \longrightarrow & \Omega^{0|s} & \xrightarrow{d} & \Omega^{1|s} & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^{n|s} & \xrightarrow{d} & \Omega^{n+1|s} & \xrightarrow{d} & \dots \\
 & \nearrow \eta & \uparrow & & \uparrow & & & & \uparrow & & \nearrow \iota & & \\
 \dots & \xrightarrow{\delta} & \Omega_{n+1|m-s} & \xrightarrow{\delta} & \Omega_{n|m-s} & \xrightarrow{\delta} & \Omega_{n-1|m-s} & \xrightarrow{\delta} & \dots & \xrightarrow{\delta} & \Omega_{0|m-s} & \longrightarrow & 0
 \end{array}$$

Here the vertical arrows are isomorphisms and the dashed arrows are defined so to make the diagram commutative. Recall from §1 that for forms as Lagrangians on paths it was necessary to introduce an augmentation. It can be shown that the “exact $0|s$ -forms” $B^{0|s}$ introduced in a formal fashion in §1 coincide with the image of $\eta : \Omega_{n+1|m-s} \rightarrow \Omega^{0|s}$.

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